

BURNSIDE PROBLEM FOR MEASURE PRESERVING GROUPS OF TORAL HOMEOMORPHISMS.

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ABSTRACT. A group G is said to be periodic if for any $g \in G$ there exists a positive integer n with $g^n = id$. We prove that a finitely generated periodic group of homeomorphisms on the 2-torus that preserves a measure μ is finite. Moreover if the group consists in homeomorphisms isotopic to the identity, then it is abelian and acts freely on \mathbb{T}^2 .

1. INTRODUCTION.

Definition 1.1. A group G is said to be **periodic** if any $g \in G$ has finite order, that is, there exists a positive integer n with $g^n = id$.

One of the oldest problem in group theory was first posed by William Burnside in 1902 (see [2]): “Let G be a finitely generated periodic group. Is G necessary a finite group?”

It is obvious that an abelian finitely generated periodic group is finite.

In 1911, Schur (see [15]) proved that this is true for subgroups of $GL(k, \mathbb{C})$, $k \in \mathbb{N}$.

But, in general, according to Golod (see [5]) the answer is negative. Adjan and Novikov (see [1]) improved this result when the orders are bounded.

Later, Ol’shanskii, Ivanov and Lysenok (see [13], [9] and [10]) exhibited many examples of infinite, finitely generated and periodic groups with bounded orders.

One of the most interesting examples is the Grigorchuk group, Γ . It is a subgroup of the automorphism group of the binary rooted tree T_2 . It is generated by four specific T_2 -automorphisms a, b, c, d , satisfying that any of the elements a, b, c, d has order 2 in Γ , therefore any element of Γ can be written as a positive word in a, b, c, d without using inverses. It has been proved (see [6]) that the Grigorchuk group is infinite, and it is a 2-group, that is, every element in Γ has finite order that is a power of 2. This shows that the Grigorchuk group is a finitely generated infinite group satisfying that every element has finite order. Moreover, this group can be realized as a subgroup of $Homeo((0, 1))$.

The problem raised by Burnside is still open for groups of homeomorphisms (or diffeomorphisms) on closed manifolds. Very few examples are known.

The following example was communicated to us by Navas:

A non trivial circle homeomorphism of finite order has no fixed points, and then a periodic group acting on a circle acts freely. Moreover, Hölder theorem states that a group acting freely on the circle is abelian (see, for example, section 2.2.4 of [12]). If it is also finitely generated then it is finite.

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Therefore, it holds that “a finitely generated periodic group of circle homeomorphisms is finite”.

Finally, we note that, even in the circle case, the hypothesis on finiteness of the generating set is crucial: the group consisting of all rational circle rotations is periodic and infinite.

Rebelo and Silva (see [14]) proved that any finitely generated periodic subgroup of C^2 -symplectomorphisms of a compact 4-dimensional symplectic manifold is finite, provided that the fundamental class in $H^4(M, \mathbb{Z})$ is a product of classes in $H^1(M, \mathbb{Z})$.

Another result in [14] is the following:

Let M be a compact (oriented) manifold whose fundamental class in $H^n(M, \mathbb{Z})$ is a product of elements in $H^1(M, \mathbb{Z})$ and whose mapping class group is finite.

Then, any finitely generated subgroup G of $\text{Diff}_\mu^2(M)$ whose elements have finite order is finite, where μ is a probability measure on M and $\text{Diff}_\mu^2(M)$ is the subgroup of orientation-preserving C^2 -diffeomorphisms of M preserving μ .

In this paper, we study a related question. We consider finitely generated periodic groups of homeomorphisms of the 2-torus, \mathbb{T}^2 . Our main result is the following:

Theorem. *A finitely generated periodic group of homeomorphisms of \mathbb{T}^2 that preserves a measure μ is finite.*

Moreover if the group consists in homeomorphisms isotopic to the identity, then it is abelian and acts freely on \mathbb{T}^2 .

Corollary 1. *An amenable finitely generated periodic group of homeomorphisms of \mathbb{T}^2 is finite and if it consists in homeomorphisms isotopic to the identity, then it is abelian and acts freely on \mathbb{T}^2 . In other words, if an amenable finitely generated periodic group acts on \mathbb{T}^2 by homeomorphisms isotopic to the identity, then the action factors through a finite abelian group which acts freely on \mathbb{T}^2 .*

Remark 1.1. *As a finite group is amenable, we get here, as a consequence of Corollary 1, another proof of lemma 4.1 of a recent paper of Franks and Handel (see [4]). It claims that “If G is a finite group which acts on \mathbb{T}^2 by homeomorphisms isotopic to the identity, then the action factors through an abelian group which acts freely on \mathbb{T}^2 .”*

Since the Grigorchuk group is amenable (see [6]) we have the following:

Corollary 2. *The Grigorchuk group can not act faithfully on \mathbb{T}^2 .*

2. PRELIMINARIES.

Definition 2.1. Let $v \in \mathbb{R}^2$, denote by $\tilde{T}_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the **translation** by v : $\tilde{T}_v(x, y) = (x, y) + v$ and by $T_v : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ its induced map on \mathbb{T}^2 and it is called **toral translation**.

A toral homeomorphism is called **pseudo-translation** if it is conjugated to a toral translation.

2.1. Homology representation.

Let $\text{Homeo}_{\mathbb{Z}^2}(\mathbb{R}^2)$ be the set of homeomorphisms $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$ and $\text{Homeo}_{\mathbb{Z}^2}^0(\mathbb{R}^2)$ be the set of homeomorphisms $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$ and $F(x + P) = F(x) + P$, for all $x \in \mathbb{R}^2$ and $P \in \mathbb{Z}^2$.

Note that a 2-torus homeomorphism is isotopic to identity if and only if its lifts belong to $\text{Homeo}_{\mathbb{Z}^2}^0(\mathbb{R}^2)$.

Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a homeomorphism and let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a lift of f . We can associate to F a linear map A_F defined by:

$$F(p + (m, n)) = F(p) + A_F(m, n), \text{ for any } m, n \text{ integers.}$$

This map A_F does not depend neither on the integers m and n nor on the lift F of f . In fact, A_F is the morphism induced by f on the first homology group of \mathbb{T}^2 . So we will denote A_f for A_F .

One can easily check the following

Properties.

- (1) *The map $A : \text{Homeo}(\mathbb{T}^2) \rightarrow GL(2, \mathbb{Z})$ defined by $A(f) = A_f$ is a morphism of groups.*
- (2) *A toral homeomorphism f is isotopic to identity if and only if $A_f = Id$.*
- (3) *Any pseudo translation is isotopic to identity.*

2.2. Measure rotation set.

In [11], rotation sets of torus homeomorphisms are introduced by Misiurewicz and Ziemian.

Definition 2.2. *Let f be a 2-torus homeomorphism isotopic to identity. We denote by \tilde{f} a lift of f to \mathbb{R}^2 . We call **measure rotation set** of \tilde{f} the subset of \mathbb{R}^2 defined by*

$$\rho_{mes}(\tilde{f}) := \{\rho_\mu(\tilde{f}) = \int_{\mathbb{T}^2} (\tilde{f} - id)d\mu, \text{ where } \mu \text{ is an } f\text{-invariant probability}\}.$$

Note that the map $\tilde{f} - id : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ is well defined, since $(\tilde{f} - id)(x + (p, q)) = (\tilde{f} - id)(x)$, for all $(p, q) \in \mathbb{Z}^2$.

Properties of the measure rotation set (see [11].)

Proposition 2.1. *Let f be a 2-torus homeomorphism isotopic to the identity and \tilde{f} be a lift of f to \mathbb{R}^2 . Let μ be an f -invariant probability and h be any 2-torus homeomorphism.*

- (1) $\rho_\mu(\tilde{f}^n) = n\rho_\mu(\tilde{f})$,
- (2) $\rho_\mu(\tilde{f} + (p, q)) = \rho_\mu(\tilde{f}) + (p, q)$, for any $(p, q) \in \mathbb{Z}^2$.

For pseudo-translations we have :

- (3) $\rho_\mu(\tilde{T}_v) = v$ for any T_v -invariant measure,
- (4) $\rho_\mu(h \circ T_v \circ h^{-1}) = A_h(v) + (p_1, q_1)$, for some $(p_1, q_1) \in \mathbb{Z}^2$, for any $h \circ T_v \circ h^{-1}$ -invariant measure. Hence, the measure rotation set of a pseudo-translation is a single vector.

Proof: The first three items are direct consequences of definitions. For last item note that exists $(p_1, q_1) \in \mathbb{Z}^2$ such that $\rho_\mu(h \circ \widetilde{T_v} \circ h^{-1}) = \rho_\mu(\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1}) + (p_1, q_1)$.

Case 1: $A_h = Id$. We prove that $\rho_\mu(\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1}) = v$.

By definition

$$\rho_\mu(\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1}) = \int_{\mathbb{T}^2} (\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1} - id) d\mu = \int_{\mathbb{T}^2} (\tilde{h} \circ \tilde{T_v} - \tilde{h}) d(h_*\mu),$$

where $h_*\mu$ is defined by $h_*\mu(B) = \mu(h(B))$ for any measurable set $B \subset \mathbb{T}^2$.

Then

$$\begin{aligned} \rho_\mu(\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1}) &= \int_{\mathbb{T}^2} (\tilde{h} \circ \tilde{T_v} - \tilde{T_v} + \tilde{T_v} - Id + Id - \tilde{h}) d(h_*\mu) = \\ &= \int_{\mathbb{T}^2} (\tilde{h} \circ \tilde{T_v} - \tilde{T_v}) d(h_*\mu) + \int_{\mathbb{T}^2} (\tilde{T_v} - Id) d(h_*\mu) + \int_{\mathbb{T}^2} (Id - \tilde{h}) d(h_*\mu). \end{aligned}$$

These integrals are well defined since h is isotopic to identity.

Since μ is $h \circ T_v \circ h^{-1}$ -invariant, the measure $h_*\mu$ is T_v -invariant. Hence,

$$\begin{aligned} \rho_\mu(\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1}) &= \int_{\mathbb{T}^2} (\tilde{h} - Id) d(h_*\mu) + \int_{\mathbb{T}^2} (\tilde{T_v} - Id) d(h_*\mu) + \int_{\mathbb{T}^2} (Id - \tilde{h}) d(h_*\mu) = \\ &= \int_{\mathbb{T}^2} (\tilde{T_v} - Id) d(h_*\mu) = v. \end{aligned}$$

Remark 2.1. Let $M \in GL(2, \mathbb{R})$, we have that $M \circ \tilde{T_v} \circ M^{-1} = \tilde{T}_{Mv}$.

Case 2: General case.

Since $A : \text{Homeo}(\mathbb{T}^2) \rightarrow GL(2, \mathbb{Z})$ is a morphism of groups, we have that

$$A_{\tilde{h} \circ A_h^{-1}} = A_{\tilde{h}} \circ A_{A_h^{-1}} = A_h \circ A_h^{-1}$$

due to linearity of A_h . Therefore $A_{\tilde{h} \circ A_h^{-1}} = Id$, so $\tilde{h} \circ A_h^{-1} \in \text{Homeo}_{\mathbb{Z}^2}^0(\mathbb{R}^2)$.

As a consequence of Remark 2.1, we have that

$$\tilde{h} \circ \tilde{T_v} \circ \tilde{h}^{-1} = \tilde{h} \circ A_h^{-1} \circ A_h \circ \tilde{T_v} \circ A_h^{-1} \circ A_h \circ \tilde{h}^{-1} = \tilde{h} \circ A_h^{-1} \circ \tilde{T}_{A_h v} \circ A_h \circ \tilde{h}^{-1}$$

By case 1, we have that $\rho_\mu(\tilde{h} \circ A_h^{-1} \circ \tilde{T}_{A_h v} \circ A_h \circ \tilde{h}^{-1}) = A_h v$.

Hence, $\rho_\mu(h \circ \widetilde{T_v} \circ h^{-1}) = A_h(v) + (p_1, q_1)$, for some $(p_1, q_1) \in \mathbb{Z}^2$. \square

Remark 2.2. Another proof of this property can be done by using the same property for the classical rotation set (see for example section 3 of [8]) and the relations between different rotation sets.

Corollary 3. If K_0 is a subgroup of $\text{Homeo}(\mathbb{T}^2)$ consisting of pseudo-translations then the **rotation map** $\rho : K_0 \rightarrow \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $\rho(k) = \rho_{mes}(\tilde{k}) \pmod{\mathbb{Z}^2}$ is well defined.

Proposition 2.2. Let K_0 be a subgroup of $\text{Homeo}(\mathbb{T}^2)$ consisting of pseudo-translations preserving a measure μ . Then the rotation map $\rho : K_0 \rightarrow \mathbb{T}^2$ is a morphism.

Proof. Let f, g in K_0 . We have to prove that $\rho(f \circ g) = \rho(f) + \rho(g)$.

By definition, $\rho(f \circ g) = \rho_{mes}(\widetilde{f \circ g}) = \rho_{mes}(\widetilde{f} \circ \widetilde{g}) \pmod{\mathbb{Z}^2}$. We have that:

$$\begin{aligned} \rho_{mes}(\widetilde{f} \circ \widetilde{g}) &:= \int_{\mathbb{T}^2} (\widetilde{f} \circ \widetilde{g}(x) - x) d\mu(x) = \int_{\mathbb{T}^2} (\widetilde{f} \circ \widetilde{g}(x) - \widetilde{g}(x)) d\mu(x) + \int_{\mathbb{T}^2} (\widetilde{g}(x) - x) d\mu(x) = \\ &= \int_{\mathbb{T}^2} (\widetilde{f}(y) - y) d(g_*^{-1}\mu)(y) + \int_{\mathbb{T}^2} (\widetilde{g}(x) - x) d\mu(x), \end{aligned}$$

with the change of variables on \mathbb{T}^2 , $y = g(x)$.

Since μ is K_0 -invariant, $g_*^{-1}\mu = \mu$ and therefore $\rho_{mes}(\widetilde{f} \circ \widetilde{g}) = \rho_{mes}(\widetilde{f}) + \rho_{mes}(\widetilde{g})$. Taking this equality $\pmod{\mathbb{Z}^2}$, we get the lemma. □

From now, we consider finitely generated periodic groups of toral homeomorphisms.

3. CLASSIFICATION OF FINITE ORDER ISOTOPIC TO IDENTITY TORAL HOMEOMORPHISMS.

In this section, we recall the classification of finite order isotopic to identity homeomorphisms of the torus up to conjugacy.

Proposition 3.1. *A finite order, isotopic to identity, toral homeomorphism is conjugated to a rational translation. In other words, it is a rational pseudo-translation.*

This result is certainly well known: A finite order toral homeomorphism is an isometry for some flat metric on \mathbb{T}^2 (see [3]). By Killing-Hopf theorem, a flat torus is isometric to an euclidian torus. Moreover, an euclidian isometry of \mathbb{R}^2 is $x \mapsto Ax + v$, with $v \in \mathbb{R}^2$, $A \in \mathcal{O}(2, \mathbb{R})$ the linear group consisting in rotations and reflections in lines.

We claim that an isotopic to identity euclidian toral isometry h is a translation. Indeed, let H be a lift of h to \mathbb{R}^2 , $H(x) = Ax + v$, and for any vector $P \in \mathbb{Z}^2$ we have that $H(x + P) = Ax + AP + v = H(x) + AP$. Since h is isotopic to identity, it follows that $H(x + P) = H(x) + P$. Therefore $A = Id$ and H is a translation.

Hence, a finite order, isotopic to identity, toral homeomorphism is conjugated to a translation by some rational vector.

A proof of this statement can also be found in [7].

Definition 3.1. *Let G be a group of toral homeomorphisms, we denoted by \mathbf{G}_0 the subgroup $\{g \in G : g \text{ is isotopic to identity}\}$.*

As a corollary of Theorem 3.1 and the remark that any pseudo-translation is isotopic to identity, we get

Corollary 4. *If G is a periodic group, then $G_0 = \{g \in G : g \text{ is a pseudo-translation}\}$ and acts freely on \mathbb{T}^2 .*

In particular, the subset of G consisting in pseudo-translations is a subgroup.

Note that a group of toral pseudo-translations always acts freely, since a non trivial pseudo-translation does not admits fix point.

4. REDUCTION TO GROUPS OF RATIONAL TORAL PSEUDO-TRANSLATIONS.

Proposition 4.1. *Let G be a finitely generated periodic subgroup of toral homeomorphisms. Then G_0 , the subgroup of G consisting in pseudo-translations, is of finite index in G .*

Proof. Denote by $A : G \rightarrow GL(2, \mathbb{Z})$ the representation of G in the homology of \mathbb{T}^2 , induced by the map A defined in section 2.1. Note that $G_0 = A^{-1}(Id)$.

As G is a finitely generated periodic group, its image by A , the group $A(G)$, is a finitely generated periodic subgroup of $GL(2, \mathbb{Z})$. Then, by Schur's result ([15]), $A(G)$ is finite.

Hence, since the quotient group $G/A^{-1}(Id)$ is isomorphic to $A(G)$, $G_0 = A^{-1}(Id)$ has finite index in G . This ends the proof. \square

As a direct consequence of this proposition we have that :

Corollary 5. *G is finite if and only if G_0 is finite.*

For proving our main theorem, it is enough proving that G_0 is finite, in fact, we will prove that G_0 is a finitely generated periodic abelian group.

5. BURNSIDE PROBLEM FOR GROUPS OF RATIONAL TORAL PSEUDO-TRANSLATIONS.

Note that a rational toral pseudo-translation is of finite order, then groups of rational toral pseudo-translations are periodic groups.

We can remark, that as in the circle case, such a group acts freely on the torus, but the Hölder theorem does not hold on the torus and the proof on the circle given in the introduction, can not be adapted to the torus.

We add the hypothesis that the rotation map is a morphism.

Proposition 5.1. *Let H_0 be a group of toral rational pseudo-translations. Suppose that the rotation map defined on H_0 is a morphism into $(\mathbb{T}^2, +)$. Then H_0 is abelian.*

Moreover, if H_0 is finitely generated then H_0 is finite.

Proof. Since the rotation map is a morphism, the commutator subgroup of H_0 consists in homeomorphisms of trivial rotation vector, indeed $\rho([f, g]) = \rho(f) + \rho(g) - \rho(f) - \rho(g) = 0$. As H_0 only contains pseudo-translations, an element g_0 of $[H_0, H_0]$ is conjugated to a translation T_v by a homeomorphism h and g_0 has trivial rotation vector. Since $(0, 0) = \rho(g_0) = A_h(v) \pmod{\mathbb{Z}^2}$, then $A_h(v) \in \mathbb{Z}^2$ so $v \in A_h^{-1}(\mathbb{Z}^2) \subset \mathbb{Z}^2$. Hence $T_v = Id$ therefore g_0 is conjugated to Id , consequently g_0 is trivial. Finally, H_0 is abelian.

As any abelian, finitely generated and periodic group is finite, H_0 is finite. \square

6. PROOF OF THE THEOREM.

By Proposition 4.1, G_0 has finite index in G . As G is finitely generated, it follows from Schreier's lemma, that states that any subgroup of finite index in a finitely generated group is finitely generated, that G_0 is finitely generated.

Moreover, according to Proposition 2.2, the rotation map ρ is a morphism.

Hence, we can use Proposition 5.1 with $H_0 = G_0$, for proving that G_0 is abelian and finite. This implies that G is finite.

Moreover, if G consists in homeomorphisms isotopic to the identity, then $G = G_0$ is abelian and consists in pseudo-translations. On the other hand, we have already noted that a group of pseudo-translations acts freely, since a pseudo-translation with a fixed point is necessarily trivial.

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